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Classics in physical geography revisited

Lorenz, E.N. 1963: Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences* 20, 130–41.[1]

This remarkable paper is a landmark in meteorology and weather forecasting, and in mathematics. The paper is also noteworthy for the clarity of its exposition. In this appraisal I give a brief description of what it contains, the significance of the results, and an outline of some recent applications of this material for the dynamics and forecasting of climate. It is first necessary to describe some mathematical details, but this is intended to be 'user-friendly', and grasping the main conclusions requires a minimum of mathematical background. Details have been skipped; if readers want all the mathematical details, they should consult the paper and other references provided.

Figure 1 Ed Lorenz on a recent hiking excursion – one of his favourite pastimes (photo courtesy of Joel Sloman)

The atmosphere may be regarded as a forced, dissipative fluid dynamical system: its motion is forced by the latitudinal imbalance of solar heating, and dissipated by friction. At the time this work was done, laboratory experiments (with a rotating annulus of fluid, cooled at the centre and heated at the outer boundary) that were designed to be a simple model of this system were known to contain a variety of types of flow. Some of these types were steady, others were periodic, and others varied, like the weather, in a manner that never seemed to repeat (eg, Hide, 1958). The equations that describe such a system must be non-linear (in the usual mathematical sense that the sum of any two solutions to them does not normally constitute a third solution). This paper describes the study of a simple but non-linear system of equations that contains such non-periodic solutions, and reveals their nature. This system is not derived from a direct analogue of the atmosphere, but its mathematical structure has the appropriate properties.

Accordingly, the body of the paper contains the description of the numerical integration of a set of non-linear equations describing thermal convection, originally due to Saltzman (1962). Specifically, they

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describe the flow of viscous, diffusive fluid between a cooled horizontal upper surface and a warmed lower one, and the full system has been truncated from a full horizontal Fourier series in the spatial structure to just one horizontal and one vertical scale of convecting cell. This leaves three variables (X, Y, Z) that vary with time and describe the amplitudes of the velocity (X) and temperature (Y) of the convecting cells, and of the horizontally averaged temperature (Z) (see Figure 2). Saltzman found that, for certain conditions, numerical integrations of the more complete equations showed that all components except these three variables decayed to zero given sufficient time. In any case, these equations themselves are the main object of interest, and they are normally written as:

$$\frac{dX}{dt} = \sigma X + \sigma Y, \qquad \frac{dY}{dt} = -XZ + rX - Y, \quad \frac{dZ}{dt} = XY - bZ. \tag{1}$$

These equations contain three constant parameters: σ, a Prandtl number (the ratio of diffusivities of momentum and heat), $r = R_a/R_c$, the Rayleigh number R_a scaled with its critical value for the onset of convection, R_c (a larger Rayleigh number implies larger thermal forcing of motion), and $b = 4/(1 + a^2)$, where a is a horizontal wavenumber for the convection cells. The subject of convection of a horizontal layer cooled from above has been well studied, and will not be pursued further here. Instead, following Lorenz, we will focus on these equations and their properties.

Equations (1) contain non-linear terms due to advection of the heated/cooled fluid by the convecting motion. They have the steady-state solution:

$$(X, Y, Z) = (0, 0, 0), \tag{2}$$

which is stable (which here means that solutions for any initial conditions eventually approach and reach this point) for r in the range $0 < r < 1$, regardless of the values of σ and b. In physical terms, this solution means that thermal diffusion is able to carry the vertical heat transport without requiring any fluid motion. In mathematical terms, the *attractor* of the system is the state of zero motion, for these conditions. When $r > 1$, there are two additional steady-state solutions:

$$(X, Y) = \pm(b(r-1))^{1/2}(1, 1), \quad Z = r - 1, \tag{3}$$

which represent steady convecting flow. These solutions are stable if:

$$r < \frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)}, \tag{4}$$

implying that they are 'attractors' of the mathematical system, but not otherwise. These two states represent steady circulation of fluid around the convection cell in one direction or its opposite, as depicted in Figure 2, and their stability implies that solutions with arbitrary initial conditions approach one or the other of these flow states. This is about as far as one can get with conventional analysis, and the study of what happens under other conditions must be pursued by numerical integration. This involves choosing a particular set of parameter values, and a particular initial condition, and integrating equations (1) forward in time.

Following the studies of Saltzman, Lorenz chose $\sigma = 10$ (a realistic value for water), and $a^2 = ½$ so that $b = 8/3$. Equation (4) then gives the criterion for instability of the steady circulations to be $r > 24.74$, and Lorenz chose to integrate equations (1) with $r = 28$, and the initial conditions (X, Y, Z) = (0, 1, 0). The evolution of the resulting solution with time may be described by the motion of a point in three-dimensional 'phase' space of coordinates (X, Y, Z), with time as a parameter marking the position along a path. The solution represented by the path of this point does not converge to a steady state, but instead tends toward a shape that may be described as resembling a three-dimensional butterfly, consisting of a pair of thin symmetrically inclined butterfly wings

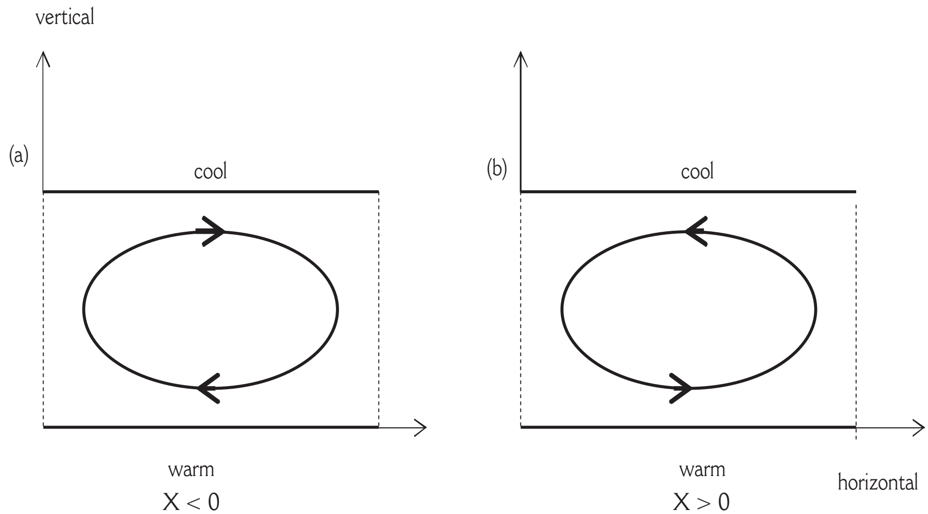

Figure 2 The Lorenz equations may be regarded as describing the temporal behaviour of flow of fluid around a single cell, cooled from above and heated from below, with slippery non-conducting side walls. X denotes the strength and direction of this circulation

that intersect at the bottom in the 'body' of the butterfly. The projection of this shape onto the X–Z plane is shown in Figure 3. The two steady-state solutions given by (3) lie near the centres of the wings, as shown in Figure 3a, but the solution never reaches these points. Instead, it cycles one or more times around this centre in one wing, until it passes into the other wing where it again cycles one or more times around the second centre, until it passes back to the first wing. This process repeats endlessly, but not in a repetitive manner. At no stage does the solution equal any given previous state, though it may become arbitrarily close to it arbitrarily often.

In terms of the variable X, the solution fluctuates with X negative for a certain period of time, and then changes to fluctuate with X positive for a time, until changing back to X negative again. Since the direction of circulation in the cell shown in Figure 2 depends on the sign of X, this means that it changes after one or more revolutions, and this must occur when the motion comes to rest, rather like a spherical pendulum coming to rest near its topmost point, after which it falls back in the reverse direction. This occurs in the 'body' of the butterfly, where X is small and the wings intersect.

The set of 'butterfly' points that the solution endlessly approaches is now termed, appropriately, a 'strange attractor'. It is not a steady, or periodic, state. Many nonlinear systems of equations are now known to have such strange attractors, and this one is known as the 'Lorenz attractor'. A projection of this object (or more accurately, a numerical solution that approximates it) is shown in Figure 3a. The strange attractor has fractal structure, which means that if one examines a small part of this set with a microscope one sees endless detail that is repeated if one zooms in on a yet smaller part of this magnified picture, and so on. Various numerical reproductions of this solution and the attractor to which it is drawn are available on a range of websites including Wikipedia, and the reader is invited to watch the development of this solution (and other solutions, choosing his/her own parameters) with time, at leisure.

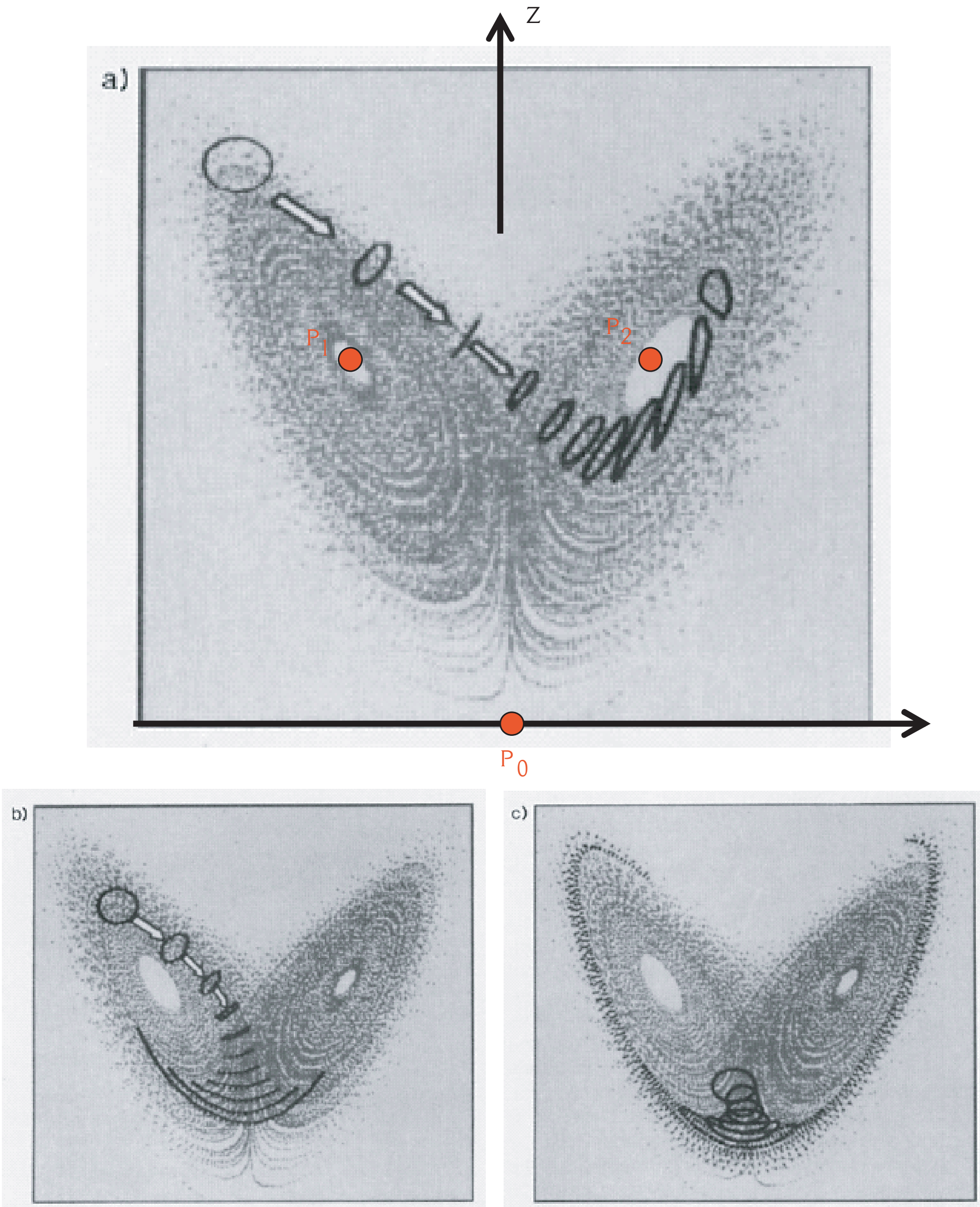

Figure 3 The Lorenz attractor, projected onto the *X–Z* plane obtained from a particular numerical integration, showing the two 'butterfly' wings or attractor basins. The points P_0, P_1 and P_2 in (a) denote the steady-state solutions described in the text: P_0 is the state of zero motion at the origin, and P_1, P_2 the two steady recirculating states. The dark lines, loops and arrows in (a), (b) and (c) are described in the text
Source: Adapted from Palmer (1993b).

It is worth emphasizing that the three steady solutions (equations 2 and 3) are all important in the structure of this 'strange attractor'. The first steady solution at the origin (equation 2) lies at the tail end of the butterfly attractor, and the curves of any non-steady solutions that pass through it denote the boundary of the attractor – the edge of the butterfly wings. The two steady convective solutions (equation 3) lie near the centre of the circulations within each wing of the attractor.

If this integration is repeated with different initial conditions, or with parameter values that are similar to but differ from those used in the above integration, the results are similar. There is no limiting end state, or cyclic behaviour. Instead the system varies continuously without repetition, cycling within one 'wing' for an extended period of time, and then within the other wing. There is no basis for predicting how many cycles there may be in each wing before the transition occurs.

In his paper, Lorenz demonstrated that the solutions drawn to this attractor are unstable. Here this means that if one takes a solution through point (X_1, Y_1, Z_1) and another through point (X_2, Y_2, Z_2), where this second point lies arbitrarily close to the first point, the two solutions may be almost identical for a while, but will eventually diverge to be totally different from and independent of each other, given sufficient time. How long this takes will depend on how close together the two initial states were. This is the essential defining property of *chaos* – that systems (or solutions representing them) lose all memory of their initial states, and are continuously unstable and sensitive to small changes or perturbations. It means that such systems are essentially unpredictable over sufficiently long periods, regardless of how well one knows and understands the physics (equations) of the system, and the predictability of the system depends on how accurately one knows the initial conditions.

After an initial decade during which it was scarcely noticed, the Lorenz system has become the most famous and most studied of all non-linear systems with strange attractors. This is because of its simplicity and the bi-stable nature of the attractor itself. The single numerical example that he described, as above, is now the subject of numerous textbooks and mathematics courses (see, for example, Sparrow, 1982; 1986; Drazin, 1992).

Lorenz applied these ideas to the problem of the predictability of weather forecasts. In the decade of the 1960s, numerical weather forecasting was developing rapidly in Princeton, USA, and Bracknell, UK, in particular, with optimism about the long-term value of this methodology. Lorenz's convective model is much simpler than any forecasting model, but the equations for the latter must be chaotic systems also, and hence possess the same unstable property. At the time, problems of adequateness of the models and accuracy of the initial observations were seen as limiting the predictability, as now, but Lorenz's results showed that there is an underlying limitation since the ideal observing system will still contain errors in the initial observations, implying an upper limit to predictability. As yet we do not know how long this is, but 10–14 days seems like a practical limit.

These equations have been used as an analogue for the weather system in other contexts. One example is the variability in the reliability of weather forecasts. Some forecasts are more reliable than others, and the Lorenz system may be used to demonstrate this as shown in Figure 3 (Palmer, 1993a). In Figure 3a, the initial state of the 'Lorenz weather' is taken to be within the dark circle at the upper left corner; integration of the equations with a variety of initial conditions within this circle constitutes an 'ensemble forecast', and gives the trajectories shown by the arrows and successive closed loops, showing that the forecast weather moves over to the other wing of the attractor, and is confined to a small area. In Figure 3b, however, integration of the equations from a different region results in a broad spread of trajectories that spans both wings, and in Figure 3c, where the initial state is near the bottom centre, the spread of points is around the perimeter of both wings. Clearly, a weather forecast becomes increasingly difficult, and more unreliable, as the initial state moves from Figure 3a to Figure 3c. Most sophisticated weather-forecasting systems now use ensemble forecasts in this manner to estimate the reliability of the forecasts obtained.

A second application of this system is to provide insight into the dynamics of climate regimes. The two wings of the Lorenz 'weather system' may be regarded as two climate regimes, and the system has two timescales: the 'weather' timescale taken to cycle around a wing, and the mean time spent in each wing before crossing over to the other regime. Palmer (1993b) has described integrations of equations (1) with weak external forcing added, representing, conceptually, forcing due to greenhouse warming, for example. The principal conclusion of this work is that the effect of the forcing does not change the nature of the climate regimes (or wings of the attractor) very much, but instead causes changes in the relative frequency in which the two regimes are occupied (these being equal in the unforced system (1)), and the relative mean length of time that the system spends in each regime.

Clearly, one must be careful not to push an analogue of this sort too far, but it is remarkable how informative and influential these studies have been, and how much has grown from Ed Lorenz's beautiful paper, which even today contains insight and points of expression that reward closer study.

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Note

1. Ed Lorenz passed away on 16 April 2008, aged 90. An enviable innings.

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